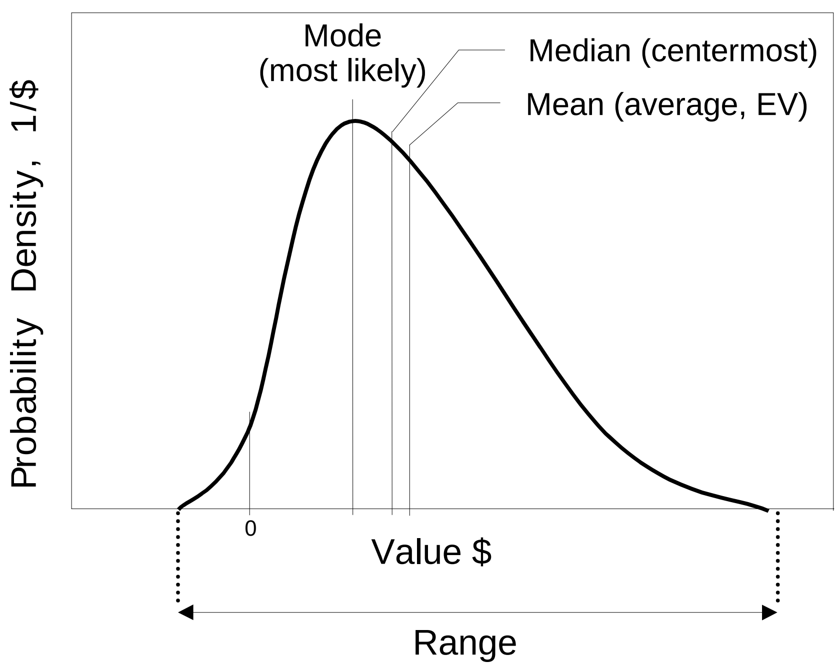
3. PROBABILITY DISTRIBUTIONS

Probability is the language of uncertainty. A ***probability distribution*** is a mathematical or graphical function that represents the likelihood of possible outcome of a ***chance event***. The variables in an evaluation model representing chance events are called ***random*** or ***stochastic variables***.

Where the possible outcomes are specific values, such as integers, we have a discrete distribution. ***Probabilities*** describe the likelihoods of discrete event outcomes. Probability is measured in unitless fractions, 1 being 100%.

Where the possible outcomes are continuous along a range there are an infinite number of possibilities. This is a continuous probability function. The graph, below, is an example. The section of the x-axis under the curve shows the range of possible outcomes for the event. The height of the function along the y-axis is proportional to the probabilities of correspond x value. In such a function, ***area*** corresponds to probability. This is formally called a ***probability density function*** (p.d.f.) because the area is analogous to mass (with density) in weighting the x-axis values. The center of mass of the distribution curve area is located exactly above the expected value.

Mean, median, and mode are all at the same value when the curve has only one peak and is symmetric.



central measures

The graph shows three popular “central measures” for a probability distribution: mode, median, and mean.

The ***mode*** is the most likely value, i.e., the value with the highest probability of occurrence. It is at the peak of the probability curve. The mode is a popular but poor measure of central tendency for most applications.

The ***median*** is the most central value. For a continuous distribution, there is equal chance of lying above and below the median. This statistic is often used when a *typical* value is desired in a highly skewed distribution, e.g., personal income or home prices. This is because the median is relatively unaffected by extremely high or low values.

Use the expected value (mean) for decision-making.

The ***mean*** is also called the *average* (ambiguous) or *expected value* (exactly synonymous). This is the most popular measure to characterize the value of a random variable. It is abbreviated μ when referring to parent populations and $\bar{x}$ when referring to a sample data set. The mean provides the best and only ***objective*** single-point value forecast. The mean may be determined in several ways.[1] One technique involves taking a distribution drawn on Cartesian (regular) graph paper and slicing the distribution vertically into segments; then, weighting representative values for the segments with the squares area under the curve in each segment.

Standard deviation is the best statistic to characterize uncertainty.

In addition to a central measure, probability distributions are often characterized by a statistic called ***standard deviation*** (σ). This is the most popular measure of risk and uncertainty. It is calculated as the square root of the variance:[2]

$$\text{variance } \sigma^2 = \sum_{\text{all } i} p(x_i)(x_i - \mu)^2 \text{ where } \quad \begin{aligned} p(x_i) &= \text{probability of } x_i \\ x_i &= \text{outcome } i \text{ value} \\ \mu &= \text{mean} \end{aligned}$$

The familiar bell-shaped curve is called a normal distribution. For a normal distribution, the following approximate confidence levels are well-established:

mean $\pm$ 1 $\sigma \cong$ 68% of the distribution's probability

mean $\pm$ 2 $\sigma \cong$ 95%

mean $\pm$ 3 $\sigma \cong$ 99.7%

As a rule-of-thumb for typical, uni-modal distributions, approximately 2/3 of the probability distribution lies in a range 2σ wide.

A characteristic feature of a decision analysis is that we represent uncertain inputs as probability distributions. Further, the analysis results can be presented to the decision-maker in the form of probability distributions. A key motivation for using decision analysis

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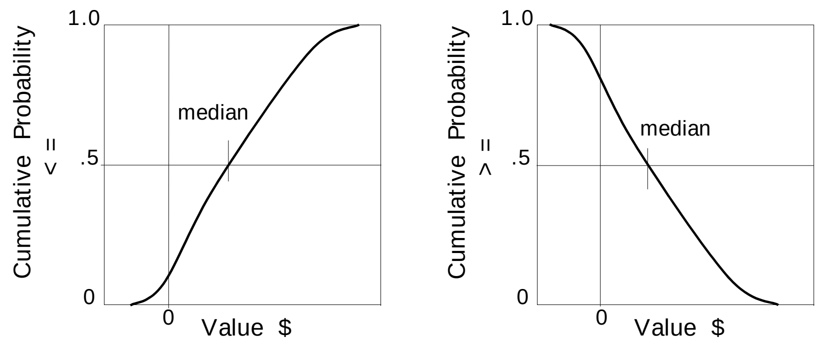
[1] Thinking of the area enclosed by the probability curve bounded by the x-axis as a 2-dimensional mass, the mean is as at the center of gravity projected to the x-axis. Area under the curve corresponds to probability, or mass in the analogy, hence the name probability density function.

[2] For the mathematically inclined, mean is the first moment about the origin, and the variance is the second moment about the mean.

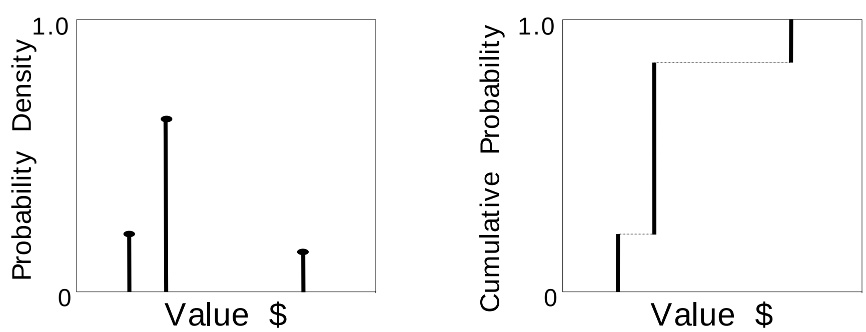
is that a probability distribution graph provides much more information than does a single-point estimate.

Cumulative Density Function

The curve on the previous page is called a density, or less accurately *frequency*, type probability curve. Another popular form, which presents exactly the same information, is the ***cumulative probability distribution***. Cumulative distributions may be either "less-than" or "greater-than (exceedance)" formats. The two forms for cumulative curves are shown below:



Where there are a finite number of possible outcomes for the random variable, we have a discrete probability function. Each possible outcome has a finite probability of occurrence. Here are example curves for a 3-level discrete distribution:



Three-level discrete level distributions are frequently used in decision trees as approximations for continuous events.